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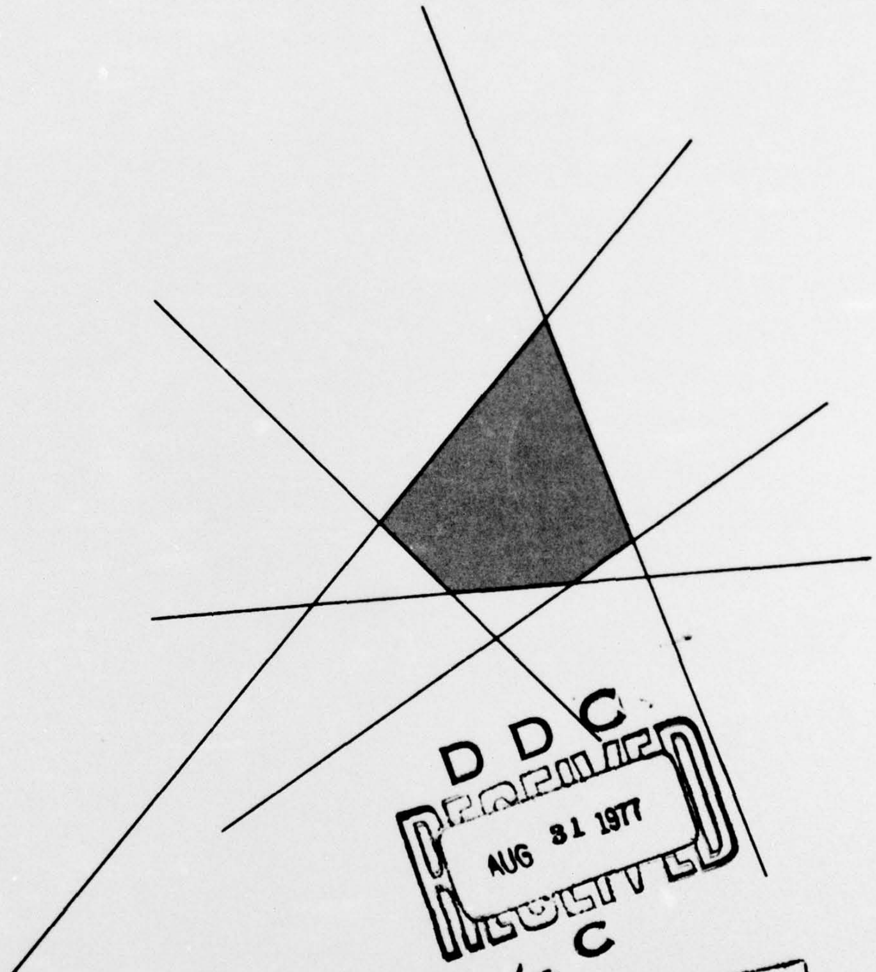
by
ROKAYA AL-AYAT
and
ROLF FÄRE



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Rokaya Al-Ayat and Rolf Färe
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University of California, Berkeley

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ABSTRACT

Within a general framework of production correspondences satisfying a set of weak axioms necessary and sufficient conditions *for the existence of a joint production function are given.* Without enforcing the strong disposability of inputs or outputs it is shown that a joint production function exists if and only if both input and output correspondences are strictly increasing along rays.

ON THE EXISTENCE OF JOINT PRODUCTION FUNCTIONS

by

Rokaya Al-Ayat and Rolf Färe

Joint production functions are frequently used in economics, however, it was not until Shephard in [6] defined such a notion within the general framework of production correspondences that its meaning became clear. The question of existence of these functions, dealt with in this paper, is yet to be settled. On this issue Shephard [8] wrote, "The joint production function is a tricky concept, seemingly simple but not shown to exist except under very restrictive conditions."

For a production technology with strongly disposable inputs and outputs Bol and Moeschlin [2], showed that continuity of both the input and the output correspondences together with *essentiality of all inputs are sufficient* for the existence of a joint production function. Later Bol in [1] showed that such a function would also exist if the essentiality condition is replaced by strict increasancy of the output correspondence in all inputs.

It is to be recalled that an output correspondence $x \to P(x) \in 2^{\mathbb{R}_+^m}$ is a mapping from input vectors $x \in \mathbb{R}_+^n$ into subsets $P(x) \in 2^{\mathbb{R}_+^m}$ of all output vectors obtainable by x. Inversely to $P(x)$ the input correspondence $u \to L(u) := \{x \mid u \in P(x)\}$ is the set of all input vectors x yielding at least an output vector u. In this paper the existence of a joint production function will be considered under the weak axioms as stated in [7]. Specifically neither the strong disposability of inputs or outputs (i.e., $x' \geqq x \in L(u) \Rightarrow x' \in L(u)$, $u' \leqq u \in P(x) \Rightarrow u' \in P(x)$ respectively) nor convexity of $P(x)$ or $L(u)$ are enforced.

Having strong disposability of inputs means that if a subvector of inputs is kept constant while the remaining are increased, output will never decrease implying there can be no congestion in the production system. In addition, strong disposability of outputs excludes their null jointness (see [9]) which is one of the basis for discussions of the external diseconomics. Thus having only weak disposability of inputs (i.e., $P(\lambda \cdot x) \supset P(x)$, $\lambda \geq 1$) and outputs (i.e., $L(\theta \cdot u) \subset L(u)$, $\theta \geq 1$) allow modelling of both congestion and null jointness.

As defined by Shephard [6], the joint production function relates input and output isoquants to each other. Recall that

$$\text{ISOQ } P(x) := \{u \mid u \in P(x) , \theta \cdot u \notin P(x) , \theta > 1\} , P(x) \neq \{0\} ,$$

and

$$\text{ISOQ } L(u) := \{x \mid x \in L(u) , \lambda \cdot x \notin L(u) , \lambda < 1\} , L(u) \not\ni \{0\} , L(u) \neq \phi .$$

Definition:

The function $F : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ such that

(1) for $u^o \geq 0$, $\text{ISOQ } L(u^o) = \{x \mid F(u^o,x) = 0\}$, $L(u^o) \neq \phi$ and

(2) for $x^o \geq 0$, $\text{ISOQ } P(x^o) = \{u \mid F(u,x^o) = 0\}$, $P(x^o) \neq \{0\}$

is a joint production function.

An equivalent statement to the definition, to be used in the sequel, was proved by Bol and Moeschlin [2] namely:

Lemma:

A joint production function $F(u,x)$ exists if and only if for all $x \geq 0$[(1)] , $P(x) \neq \{0\}$ and $u \geq 0$, $L(u) \neq \phi$, $u \in \text{ISOQ } P(x) \Leftrightarrow x \in \text{ISOQ } L(u)$.

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(1) $x \geq 0$ means $x \geqq 0$ but $x \neq 0$.

Theorem:

For all $x \geq 0$, $u \geq 0$ such that $P(x) \neq \{0\}$, $L(u) \neq \phi$ with $x \to P(x)$ $(u \to L(u))$ satisfying the weak axioms, a necessary and sufficient condition for the existence of a joint production function $F(u,x)$ is

$$(*) \quad \text{ISOQ } P(x) \cap \text{ISOQ } P(\lambda \cdot x) = \text{ISOQ } L(u) \cap \text{ISOQ } L(\theta \cdot u) \quad \text{empty}$$

for all positive scalars λ , $\theta \neq 1$.

Proof:

To show the necessity of (*) , assume there is a joint production function $F(u,x)$ and let $u \in \text{ISOQ } P(x) \cap \text{ISOQ } P(\lambda \cdot x)$. By the lemma, $x \in \text{ISOQ } L(u)$ and $\lambda \cdot x \in \text{ISOQ } L(u)$, $\lambda \neq 1$, which is a contradiction. Thus if a joint production exists, $\text{ISOQ } P(x) \cap \text{ISOQ } P(\lambda \cdot x)$ is empty for all positive scalars λ , $\lambda \neq 1$. A similar argument can be used to show that the existence of $F(u,x)$ implies that for all positive θ , $\theta \neq 1$, $\text{ISOQ } L(u) \cap \text{ISOQ } L(\theta \cdot u)$ is empty.

To show the sufficiency, assume that (*) holds, and that for $x \geq 0$, $P(x) \neq \{0\}$, $u \in \text{ISOQ } P(x)$ but $x \notin \text{ISOQ } L(u)$. From the definition of the isoquant, there exists a $\lambda < 1$ such that $\lambda \cdot x \in \text{ISOQ } L(u)$ implying that $u \in P(\lambda \cdot x)$. But from the weak disposability of inputs $P(\lambda \cdot x) \subset P(x)$ which together with (*) implies that $u \notin \text{ISOQ } P(x)$, a contradiction. Similarly it can be shown that having $\text{ISOQ } L(u) \cap \text{ISOQ } L(\theta \cdot u)$ empty would guarantee that $x \in \text{ISOQ } L(u) \Rightarrow u \in \text{ISOQ } P(x)$. Hence the sufficiency of (*) for the existence of a joint production function is proved. See lemma. Q.E.D.

Continuity of the production correspondences has not been enforced. However, following an argument similar to that used by Bol and Moeschlin in [2] one can prove:

Corollary:

If a joint production function exists, then both the input and the output correspondences are continuous along rays i.e., $P(\lambda^o \cdot x) = \overline{\bigcup_{0<\lambda<\lambda^o} P(\lambda \cdot x)}$ and $L(\theta^o \cdot u) = \overline{\bigcup_{\theta>\theta^o} L(\theta \cdot u)}$ respectively, with $u, x \neq 0$.

Note that continuity along rays together with strong disposability imply continuity (see [2] for definition).

Next, consider the production technology;

$$P(x_1,x_2) := \{\{(u_1,0)\} \cup \{(0,u_2)\} \mid 0 \leqq u_i \leqq x_i \ , \ i = 1,2\}$$

and inversely

$$L(u_1,u_2) := \{\{(x_1,0)\} \cup \{(0,x_2)\} \mid x_i \geqq u_i \ , \ i = 1,2\} .$$

The corresponding isoquants are given by

$$\text{ISOQ } L(u_1,u_2) = \{\{(x_1,0)\} \cup \{(0,x_2)\} \mid x_i = u_i \ , \ i = 1,2\}$$

and

$$\text{ISOQ } P(x_1,x_2) = \{\{(u_1,0)\} \cup \{(0,u_2)\} \mid u_i = x_i \ , \ i = 1,2\} .$$

In this example, the production correspondence satisfies the weak axioms, but neither strong disposability of inputs and outputs nor the essentiality condition (i.e., $P(x) \neq \{0\}$ implies $(x_1,x_2) > (0,0)$) used in [2] hold. Yet it is clear that a joint production function exist.

Finally, an example not satisfying the sufficiency conditions applied in [1] and [2] is given. Before introducting it the following proposition to be used, is proved.

Proposition:

If the production function $\phi(x) := \max \{u \mid x \in L(u)\}$, is continuous and strictly increasing along rays in the input space $\mathbb{R}^n_+$, ISOQ $L(u) = \{x \mid \phi(x) = u\}$, $u > 0$.

Proof:

Clearly ISOQ $L(u) \subset \{x \mid \phi(x) \geqq u\}$, $u > 0$; let $x^o \in \{x \mid \phi(x) > u\}$. Since ϕ is continuous along rays, $\{\lambda \mid \phi(\lambda \cdot x^o) > u\}$ is open implying that $x^o \notin$ ISOQ $L(u)$, hence ISOQ $L(u) \subset \{x \mid \phi(x) = u\}$. Next assume $x^o \notin$ ISOQ $L(u)$, $u > 0$, then since ϕ is strictly increasing along rays, if $x^o \in L(u)$, there is a $\lambda < 1$ such that $\phi(\lambda \cdot x^o) = u$ implying that $x^o \notin \{x \mid \phi(x) = u\}$. Q.E.D.

Now, consider the output correspondence $x \to P(x) \subset [0,+\infty)$,

$$P(x) := \left\{u \;\middle|\; 0 \leqq u \leqq A \cdot \left[(1-\delta)\cdot \max\left\{0,(x_1 - \gamma \cdot x_2)^{-\rho}\right\} + \delta \cdot x_2^{-\rho}\right]^{-1/\rho} =: \phi(x)\right\}$$

where the parameters of the WDI - production function $\phi(x)$ are $A > 0$, $\delta \in (0,1)$, $\gamma \in (0,\infty)$ and $\rho \in (-1,0)$ (see [3]). For these values of the parameters, $\phi(x)$ is upper semi-continuous which is equivalent to $P(x)$ being upper hemi-continuous (see [5], p. 22) also $x_2 = 0$ does not imply $P(x) = \{0\}$ and ϕ is not increasing in x_2 . Thus $P(x)$ does not meet the continuity requirement of [1] and [2] nor does it meet the other sufficiency condition of [2] (essentiality of all factors) or [1] (strict increasancy in all factors).

Using the proposition above the isoquants of $P(x)$ and $L(u)$ are easily computed to be,

$$\text{ISOQ } P(x) = \{u \mid u = \phi(x)\} \quad \text{and} \quad \text{ISOQ } L(u) = \{x \mid \phi(x) = u\} .$$

Thus, $x \in \text{ISOQ } L(u) \Leftrightarrow u \in \text{ISOQ } P(x)$, showing that under the weak axioms for a production technology, the sufficient conditions found in [1] and [2] need not hold for a joint production function to exist.

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